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Some exact results for moments of 2D directed animals

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Abstract. Using computer enumerations and the algebraic approximant method of series analysis, several new exact results have been found for moments of width, perimeter and loops for directed animals on the square and triangular lattices. A proof using q -series is given for exact solutions for widths on both the square and triangular lattices.

Exact properties of animals (connected clusters of sites on some lattice) are poorly understood in even two dimensions. Directed animals (e.g. figure 1) are better understood. They add the restriction that each site other than a given base site is in a preferred direction of some site on the animal. Dhar *et al* [1] used a computer to enumerate the total number of directed animals on the square, triangular and honeycomb lattices, and discovered a simple expression for the generating function on the square and triangular lattices which was later proved [2] by an equivalence with the hard square lattice gas problem, followed by other proofs (e.g. [3–6]). There has also been largely unsuccessful work to find exact results on the honeycomb and other lattices, and for bond animals and varieties of trees. A solution on a variety of the decorated lattice was found in [7] and proved in [8].

As well as counting the number of these animals by sites, other properties are interesting. Duarte [9, 10] looked at the average perimeter, several people looked at percolation (e.g. [11, 12]) and longitudinal [13–16] and lateral [13, 15, 17] radii of gyration.

In this paper the series for the moments of various properties have been significantly extended on the square and triangular lattices, and some exact results have been found. The algorithm used is a generalization of the algorithm used in [7] which was based on the dynamic programming algorithm in [1]. The series were analysed with algebraic [18] and differential [19] approximants.

1. Nomenclature

In this paper, variables for which the moments refer to per-site values are lower case letters; variables for which the moments refer to animal values are upper case letters. A superscript in square brackets gives the moment in question. If a subscript n is present, the variable refers to the *mean* over all animals with n sites. If no subscript is present, one is referring to the quantity evaluated for a particular animal. The total number of animals with n sites is c_n . One then uses W for width, L for length, H for number of loops, and P for perimeter.

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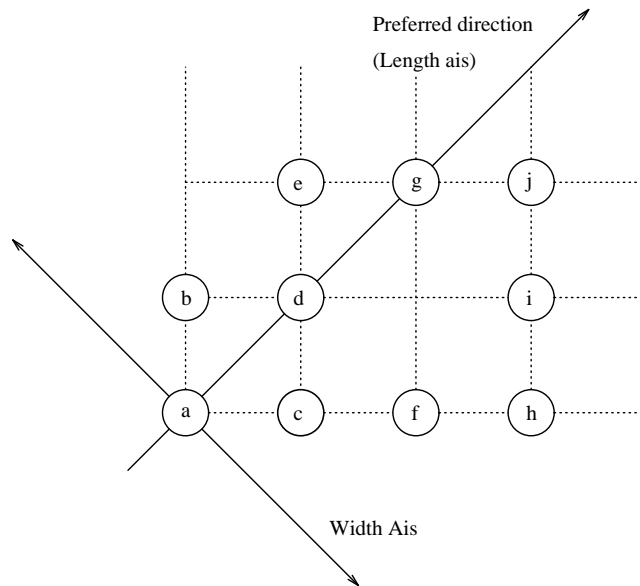


Figure 1. An example directed animal on a square lattice (broken lines) with the length (preferred direction) and width axes marked. For clarity, unoccupied sites are not drawn.

Using this notation, if \mathcal{A} is an animal, s is a site, and $\text{width}(s)$ is the width of the single site s ,

$$W^{[m]}(\mathcal{A}) = \left(\sum_{s \in \mathcal{A}} \text{width}(s) \right)^m \quad (1)$$

$$w^{[m]}(\mathcal{A}) = \sum_{s \in \mathcal{A}} (\text{width}(s))^m \quad (2)$$

$$w_n^{[m]} = \frac{1}{c_n} \sum_{|\mathcal{A}|=n} w^{[m]}(\mathcal{A}) \quad (3)$$

$$W_n^{[m]} = \frac{1}{c_n} \sum_{|\mathcal{A}|=n} W^{[m]}(\mathcal{A}). \quad (4)$$

Trivially, $w^{[1]} = W^{[1]}$ for all properties.

In figure 1, the given animal has the following properties.

- Area (number of sites) is $n = 10$.
- Perimeter (number of unoccupied sites where one could add a site without breaking the directed animal requirements) is $P^{[1]} = 8$ (above b, e, g, j, f , and to the right of j, i , and h).
- Squared perimeter is $P^{[2]} = 8^2 = 64$.
- Number of loops is $H^{[1]} = 2$ (d and j).
- Squared number of loops is $H^{[2]} = 2^2 = 4$.
- Length $l^{[1]} = L^{[1]} = 25$ (0 from a , 1 from b and c , 2 from d and f , 3 from e and h , 4 from g and i and 5 from j).
- Squared length is $l^{[2]} = 85$ (the sum of the squares of the previous contributions).
- Squared animal length is $L^{[2]} = 25^2 = 625$.
- Width is $w^{[1]} = W^{[1]} = 7$ (-1 from b and e , 0 from a, d and g , 1 from j and c , 2

from i and f , and 3 from h .

- Squared width is $w^{[2]} = 21$ (1 from b and e , 0 from a , d and g , 1 from j and c , 4 from i and f , and 9 from h).

- Squared animal width is $W^{[2]} = 7^2 = 49$.

1.1. Moments of inertia

For all symmetric lattices (for instance the square and triangular lattices, but not the hexagonal lattice, as it is not symmetric when put on the square lattice with a preferred direction), the average width $w_n^{[1]}$ is always zero (as are all odd moments), and is thus not very interesting. A second moment of width is thus more interesting.

The definitions of width and length used here are slightly different quantities to those normally studied, such as [15, 13, 20, 14], in which the authors look at the overall moments of inertia I_{\parallel} and I_{\perp} of the animals around their centre of mass, or equivalently their radii of gyration R_{\parallel} and R_{\perp} . The inertia moments and radii of gyration are related, as usual, by $I = nR^2$. It is found that $R_{\perp} \approx n^{\nu_{\perp}}$ and $R_{\parallel} \approx n^{\nu_{\parallel}}$ with $\nu_{\perp} \approx \frac{1}{2}$ and $\nu_{\parallel} \approx \frac{8}{11}$ for large n . These results mainly come from numerical work from series enumerations [15, 13, 20, 14], with an exact result of $\nu_{\perp} = \frac{1}{2}$ coming from field theory [17].

For undirected animals, these are the only objects of possible interest. However, for directed animals one has a second reference point, the origin, about which one can also calculate moments of inertia. This is my reference point, as it makes the discussion of the algorithm significantly easier. The two reference points can be related easily:

$$I_{\parallel} = l^{[2]} - L^{[2]}/n \quad (5)$$

$$I_{\perp} = w^{[2]} - W^{[2]}/n. \quad (6)$$

2. Algorithm

In [7] the basic algorithm used for enumerating directed animals is described. It is basically a dynamic programming algorithm that calculates the number of animals $A(d, s, n)$ starting from a base consisting of a diagonal line perpendicular to the preferred direction with sites occupied according to a binary ‘signature’ s with n sites (or whatever enumeration variable is desired) to be added in the preferred direction from the diagonal, and starting at state d . The state d is used to embed lattices such as the honeycomb on the square lattice. This diagonal is hereafter referred to as the *main diagonal*.

Every $A(d, s, n)$ with $n > 0$ can then be represented as a linear combination of other $A(\delta, \sigma, \nu)$ values, with $\nu < n$. When n reaches zero, the value $A(d, s, 0)$ is usually trivially 1. By storing intermediate values rather than recalculating, this algorithm becomes very efficient.

For instance, on the square lattice (where there are no states, so the first variable d has been omitted for clarity),

$$A(1, n) = A(1, n - 1) + A(2, n - 1) + A(3, n - 2). \quad (7)$$

This represents a signature 1 (one site) causing there to be three options for the sites occupied on the next diagonal perpendicular to the preferred direction. These three options are either one site to the right, one site above, or both to the right and above sites. Various tricks such as noting that $A(2, n) = A(1, n)$ are then used to further increase efficiency. This is referred to later as ‘shifting down to normalize’.

This algorithm was used and described in [7] and [21] to enumerate animals on a variety of lattices by number of sites or bonds, allowing or disallowing loops. It has also been used

with slight modifications to enumerate by both site and bond perimeter, and could also be used for high and low density percolation.

In order to calculate moments as well, the same basic approach is used. The difference is that in this case the information about each sub-animal, previously denoted by the single value $A(d, s, n)$, now must also contain information about the moments for the animals enumerated in A . Let us say that two moments are required in addition to the number of animals. Call these moments M_1 and M_2 , respectively.

There are four properties that were studied here:

- The width (signed distance perpendicular to the preferred direction).
- The length (distance parallel to the preferred direction).
- The perimeter if one is enumerating by area, or the area if one is enumerating by perimeter.
- The number of site loops.

No exact generating functions were found for length or area when enumerating by perimeter.

If one uses units of $1/\sqrt{2}$ of the lattice spacing, then all sites have integer widths and lengths on the square lattice. On the triangular lattice, units of $\sqrt{3}/2$ for width and $\frac{1}{2}$ for length are used. For the purposes of this paper, the triangular lattice has been impressed upon the square lattice in the same manner as [7], and thus can be considered to be a square lattice with extra diagonal bonds joining each site to the site one above and to the right. The preferred direction then allows animals to grow upwards, to the right, or to the upper right. The analogue on the normal triangular lattice is to have the preferred direction allow growth along three consecutive bonds from each site. The generating functions here are not changed by the geometry used, although the interpretation of the units of width is slightly different, accounting for the different units just mentioned.

One may define the width and length in many ways, of which two are perhaps the most sensible: by maximum width/length for the animal and by average by site. A site average definition is usually concerned with a radius of gyration. Determining the maximum width of an animal is difficult if not impossible using the above algorithm, whereas the average is straightforward. Using the average width and length is also convenient as it means that the first moment is the centre of gravity, and the second moment is the radius of gyration. These have strong physical significance. Note that for symmetrical animals the first moment of the width should be trivially zero. There are also two ways of considering moments of width and length: individually by site, or by whole animal.

More generally, there are two separate classes of properties: those that are properties of the animal in its entirety (area, perimeter, site loops, and the 'whole animal' variety of width and length), and others that are properties of individual sites (width, length).

Consider now the first class: those that are properties of the animal in its entirety. Let

$$M_m = \sum_{a \in A} P^m(a) \quad (8)$$

where A is now the set of all animals counted in the scalar A above, and $P(a)$ is a scalar property of the (sub)animal a †. One now wants to obtain an equivalent to the recurrence relation (7). Suppose that we are building up some set of animals A and we use some sets of animals \bar{A}_i in the process. Then the analogue of equation (7) for A is still

$$|A| = \sum_i |\bar{A}_i|. \quad (9)$$

† P is any property, and should not be confused with P in section 1 which refers solely to the property of perimeter.

A similar equation for the moments follows:

$$M_m = \sum_i \sum_{a \in A_i} P^m(a + s) \quad (10)$$

where s are the extra sites added at the main diagonal. The form (and indeed existence) of this recurrence relation depends on the definition of $P(a + s)$. In order for this method to work, it must be possible to apply the method of recursive division to the moments of the properties being studied as well as the previously done number of animals. There are two steps in the recursive breakdown:

- Breaking the animal down into a signature line on the main diagonal (s) and the rest of the animal ($a \in A_i$) which is described recursively. To make this practical, we want to be able to say that $P(s + a) = P(a) + \Delta_s$.

- Shifting the animal one step in the preferred direction and the same number of steps down the main diagonal to make the dynamic programming more efficient. To make this practical we want to be able to say that $P(S(a)) = P(a) + \Delta_S |a|$.

The number of site loops in a directed animal is very easy to determine by looking at each site individually. Each site in a directed animal must be to an allowed direction of at least one occupied site—the site it ‘grew’ from. If it were possible for it to grow from more than one site, then this implies that there is a loop. On the square lattice this means both of the preceding sites must be occupied; on the triangular lattice it means at least two must be occupied. From this perspective it is reasonable to have two definitions of loops on the triangular lattice; those with two occupied previous neighbours, and those with all three occupied previous neighbours. More conventional definitions (such as the cyclomatic index) can be easily obtained as some linear combination of these two definitions. Hereafter I have referred to the former definition as *common* loops and the latter as *uncommon* loops. As a result of this site-wide property, one can say that Δ_S is just the number of loop sites in the signature s . The number of loops is not affected by shifting, so $\Delta_S = 0$.

Area is simpler still; $\Delta_s = |s|$ and $\Delta_S = 0$.

For length, $\Delta_s = |s|$ representing the moments of the $|s|$ sites which are on the current main diagonal, and thus have a length (distance in the preferred direction which is perpendicular to the main diagonal) of exactly one each. The shift of the rest of the animal a in the direction of the main diagonal of one unit means that $\Delta_S = 1$ to make up for the future undercounting of the rest of the n_i sites of each animal a in A_i . Length is not affected by a transverse shift.

Perimeter is slightly harder to explain, as the most computationally reasonable method of doing it is to define the perimeter of an animal portion to include those sites on the main diagonal that could be occupied due to being able to grow out of previous sites, but happen to not be in the current signature. With this definition then Δ_s is the number of such sites in the current signature, and $\Delta_S = 0$.

For width, Δ_s is the moment of width for each site in the signature s , with the origin (zero width point) being the lowest occupied site on the main diagonal. Width is affected by a shifting along the main diagonal. If the animal is not explicitly shifted down, then the width origin has effectively moved $1/\sqrt{2}$ times the lattice spacing as it grows horizontally, so $\Delta_S = -1$ using the unit system previously described. For every site the animal slides down the main diagonal, the width origin has moved by a distance of $\sqrt{2}$ times the lattice spacing, so Δ_S must be increased by 2. Thus Δ_S is twice the number of times shifted, minus one.

Then if we define $\delta = \Delta_s + n_i \Delta_S$ equation (10) becomes

$$M_m = \sum_i \sum_{a \in \bar{A}_i} (P(a) + \delta)^m \tag{11}$$

whence

$$M_1 = \sum_i \bar{M}_{1i} + \delta |\bar{A}_i| \tag{12}$$

$$M_2 = \sum_i \bar{M}_{2i} + 2\delta \bar{M}_{1i} + \delta^2 |\bar{A}_i|. \tag{13}$$

A similar process with slightly different equations and variables works when one considers properties depending on each site individually rather than as part of an animal, such as for the width and length when the moments are defined by

$$M_m = \sum_{a \in A} \sum_{l \in a} P^m(l) \tag{14}$$

where l are the single sites (leaves) in a given animal a . Then, a recurrence relation like (11) will be

$$M_m = \sum_i \sum_{a \in \bar{A}_i} \sum_{l \in S(a)+s} P^m(l) \tag{15}$$

$$= \sum_i \left(|\bar{A}_i| \sum_{l \in s} P^m(l) + \sum_{a \in \bar{A}_i} \sum_{l \in a} P^m(S(l)) \right) \tag{16}$$

where $S(a)$ is the animal formed by shifting a as required by the recurrence relation. Again, we would like $P(S(l)) = P(l) + \Delta_s$ as defined previously. Define γ_m to be $\sum_{l \in s} P^m(l)$. Note that γ_1 is the same as Δ_s as defined previously. Then

$$M_1 = \sum_i (|\bar{A}_i| [\gamma_1 + n_i \Delta_S] + \bar{M}_{1i}) \tag{17}$$

$$M_2 = \sum_i (|\bar{A}_i| [\gamma_2 + n_i \Delta_S^2] + \bar{M}_{2i} + 2\Delta_S \bar{M}_{1i}) \tag{18}$$

where n_i is the number of sites in each of the animals in set \bar{A}_i .

These recurrence relations can then be used in the same way as the previous simpler relations involving just A . Note that the memory requirement is increased only linearly in the number of moments, as there are now three numbers ($|A_i|, M_{1i}, M_{2i}$) to keep track of where there was previously only one.

3. Results

After calculating the series for these moments, they were analysed via the method of algebraic approximants [18] and in some cases produced an exact algebraic solution to the generating function. These exact solutions are given below, together with an asymptotic expansion for the moment (per animal).

The vast majority of series obtained did not have an easily discoverable algebraic generating function. Some of these series have been analysed by differential approximants and other methods [22]. They are not included here.

Note that the following solutions come from empirical observations that the first 40 terms (for the square lattice) and the first 26 terms (for the triangular lattice) satisfy the given equations. Since these approximants are generated from far fewer terms, it is exceedingly unlikely that they are incorrect, but this is not by any means a formal proof.

Each of the generating functions here is presented as a quadratic, which of course gives two different generating functions. There is no difficulty in choosing the correct solution, however, as the non-physical solution is usually obviously wrong (for instance, being negative inside the radius of convergence).

The asymptotic values are then obtained by breaking the generating function into a linear combination of terms like $f(x) = \sum_{\alpha} C_{\alpha}(1 - \mu x)^{-\alpha}$ where $1/\mu$ is the critical point, then using the relation that if

$$\sum_{n=0}^{\infty} a_n x^n = (1 - \mu x)^{-\alpha} \tag{19}$$

then from the binomial theorem and equation (6.1.47) of [23],

$$a_n = \mu^n \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left[1 + \frac{\alpha(\alpha - 1)}{2n} + \frac{\alpha(\alpha - 1)(\alpha - 2)(3\alpha - 1)}{24n^2} + O(n^{-3}) \right]. \tag{20}$$

For a given lattice topology, the critical point must be the same for all moments considered, as it is clear that each moment (per animal) has a polynomial upper bound. For instance, for an animal of size n sites on the square lattice, the perimeter will clearly be bounded by $n + 1$, the number of loops by n , and the width and length by n . The higher order moments will be bounded by a similar polynomial. For directed animals on the square lattice, $\mu = 3$; for directed animals on the triangular lattice, $\mu = 4$ [1]. These asymptotic expressions can then be divided by the asymptotic expression for the number of animals to get an asymptotic values for the moment per animal.

The results are given in table 1 for the square lattice and in table 2 for the triangular lattice, giving for each moment the generating function, the asymptotic expansion and the per animal asymptotic expansion. The number of directed animals is also given for reference.

4. Comments

The result for the average perimeter on the square lattice confirms Duarte’s conjecture [9, 10] that the average perimeter was $3n/4 + o(1)$. It is interesting in that it indicates that the perimeter of the animal grows linearly with size, in a manner characteristic of one-dimensional, hole ridden, or spread out objects. This may be compared to holeless compact two-dimensional objects, where the perimeter grows characteristically like \sqrt{n} .

The expression for R_{\perp} on the square lattice confirms $v_{\perp} = \frac{1}{2}$ as expected in [13, 15] and proved in [17], and the expectation in [14] that the first correction term for R_{\perp} is analytic is confirmed by the exact value $(-5/16n)$. As expected by universality, $v_{\perp} = \frac{1}{2}$ also on the triangular lattice. It is interesting that the exact formula for R_{\perp} on the triangular lattice is so simple. These width formulae, though originally derived empirically, are proved using q -series and generalized q -series later in this paper.

From the expression for the average and the average squared number of loops on the square lattice, one can get the variance for the number of loops $H^{[1]}$ in an animal of n sites:

$$\sigma^2 = \frac{7n}{54} \left(1 - \frac{375}{8n} + O(n^{-2}) \right). \tag{21}$$

5. Proving these exact results

There are many new results in tables 1 and 2 that may be analytically proved, and should be easier to do so now that the answer is known. I have only proved the four width equations

Table 1. Moments for directed animals on the square lattice.

Moment	Generating function f	Expansion	Per animal
Number	$(3x - 1)f^2 + (3x - 1)f$ $+x = 0$	$c_n = 3^n \frac{1}{\sqrt{3\pi}} n^{-\frac{1}{2}}$ $(1 - \frac{1}{16n} + \frac{13}{512n^2} + O(n^{-3}))$	1
Perimeter	$x(x + 1)^3(3x - 1)^3 f^2$ $+(x^2 + x + 1)$ $\times (x + 1)^2(3x - 1)^3 f$ $+x(2 - 6x - 5x^2 + 12x^3$ $+13x^4 + 12x^5 + 9x^6) = 0$	$P_n^{[1]} c_n = \frac{3}{4} 3^n \frac{1}{\sqrt{3\pi}} n^{\frac{1}{2}}$ $(1 + \frac{115}{48n} - \frac{2645}{1536n^2} + O(n^{-3}))$	$P_n^{[1]} = \frac{3}{4} n$ $(1 + \frac{59}{24n} - \frac{153}{96n^2} + O(n^{-3}))$
Width (inertia)	$(x + 1)(3x - 1)^5 f^2$ $+4x^4 = 0$	$w_n^{[2]} c_n = \frac{4}{9} 3^n \frac{1}{\sqrt{3\pi}} n^{\frac{3}{2}}$ $(1 - \frac{15}{16n} + \frac{17}{512n^2} + O(n^{-3}))$	$w_n^{[2]} = \frac{4}{9} n^2$ $(1 - \frac{7}{8n} - \frac{3}{64n^2} + O(n^{-3}))$
Width (centroid)	$(x + 1)^3(3x - 1)^7 f^2$ $+4x^4(1 + x + 3x^2)^2 = 0$	$W_n^{[2]} c_n = \frac{2}{9} 3^n \frac{1}{\sqrt{3\pi}} n^{\frac{5}{2}}$ $(1 - \frac{9}{16n} + \frac{281}{512n^2} + O(n^{-3}))$	$W_n^{[2]} = \frac{2}{9} n^3$ $(1 - \frac{9}{8n} + \frac{29}{64n^2} + O(n^{-3}))$
Inertia			$I_{\perp} = \frac{2}{9} n^2$ $(1 - \frac{5}{8n} - \frac{35}{64n^2} + O(n^{-3}))$
Radius of gyration			$R_{\perp} = \frac{\sqrt{2}}{3} \sqrt{n}$ $(1 - \frac{5}{16n} - \frac{305}{512n^2} + O(n^{-3}))$
Loops	$(x + 1)(3x - 1)^3 f^2$ $-(x + 1)(3x - 1)^3 f$ $+x^4(4x^2 + 2x - 1) = 0$	$H_n^{[1]} c_n = \frac{1}{9} 3^n \frac{1}{\sqrt{3\pi}} n^{\frac{1}{2}}$ $(1 - \frac{65}{16n} + \frac{1357}{512n^2} + O(n^{-3}))$	$H_n^{[1]} = \frac{1}{9} n$ $(1 - \frac{4}{n} + \frac{19}{8n^2} + O(n^{-3}))$
Squared loops	$(x + 1)^3(3x - 1)^5 f^2$ $+(x - 1)(2x^2 + 4x + 1)$ $\times (x + 1)^2(3x - 1)^5 f$ $+x^4(-1 + 4x + 10x^2 - 52x^3$ $+27x^4 + 48x^5 - 108x^6$ $+144x^7 + 252x^8) = 0$	$H_n^{[2]} c_n = \frac{1}{81} 3^n \frac{1}{\sqrt{3\pi}} n^{\frac{3}{2}}$ $(1 + \frac{39}{16n} - \frac{17443}{512n^2} + O(n^{-3}))$	$H_n^{[2]} = \frac{1}{81} n^2$ $(1 + \frac{5}{2n} - \frac{543}{16n^2} + O(n^{-3}))$

so far. There are many ways that have been used to solve such equations analytically, such as

- bijections to algebraic-type languages (DSV) and related q series [6, 24, 3, 5, 4, 25];
- equivalence to other physical or mathematical systems [2, 26–28];
- the Temperley method (splitting up the generating function into smaller generating functions for which one can write a recurrence relation) [29–32];
- analytical transfer matrices [13, 20];
- heaps of pieces (a bijective type approach) [24, 33].

A much longer and more detailed list is given in [34].

The method of a bijection to a *gingois*[†] tree [6] works in a straightforward manner for proving the series associated with width, as the bijection between directed animals and *gingois* trees in [6] preserves width, though not length. This can be solved using a technique similar to q -series, except with more than one ‘ q ’-type variable. One is in principle then able

[†] French for slightly asymmetric or skew-whiff.

Table 2. Moments for directed animals on the triangular lattice.

Moment	Generating function f	Expansion	Per animal
Number	$(4x - 1)(f^2 + f) + x = 0$	$c_n = 4^n \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{2}}$ $(1 - \frac{1}{8n} + \frac{1}{128n^2} + O(n^{-3}))$	1
Width (inertia)	$(4x - 1)^5 f^2 + 4x^4 = 0$	$w_n^{[2]} c_n = \frac{1}{3} 4^n \frac{1}{2\sqrt{\pi}} n^{\frac{3}{2}}$ $(1 - \frac{9}{8n} + \frac{17}{128n^2} + O(n^{-3}))$	$w_n^{[2]} = \frac{1}{3} n(n - 1)$
Width (centroid)	$(4x - 1)^7 f^2 + 4x^4(x + 1)^2 = 0$	$W_n^{[2]} c_n = \frac{1}{6} 4^n \frac{1}{2\sqrt{\pi}} n^{\frac{5}{2}}$ $(1 - \frac{9}{8n} + \frac{17}{128n^2} + O(n^{-3}))$	$W_n^{[2]} = \frac{1}{6} n^2(n - 1)$
Inertia			$I_{\perp} = \frac{1}{6} n(n - 1)$
Radius of gyration			$R_{\perp} = \frac{1}{\sqrt{6}} \sqrt{n - 1}$
Loops (common)	$(4x - 1)^3 f^2 - (4x - 1)^3 f + (9x - 2)x^3 = 0$	$H_n^{[1]} c_n = \frac{1}{4} 4^n \frac{1}{2\sqrt{\pi}} n^{\frac{1}{2}}$ $(1 - \frac{21}{8n} + \frac{137}{128n^2} + O(n^{-3}))$	$H_n^{[1]} = \frac{1}{4} n$ $(1 - \frac{5}{2n} + \frac{3}{4n^2} + O(n^{-3}))$

to get an expression for the generating function in terms of three variables, whose powers represent n , the number of sites, $W_n^{[1]}$, the average width, and $w_n^{[2]}$, the moment of inertia. It is difficult to get an explicit expression to the coupled set of equations produced, though with partial differentiation, it is straightforward (though rather complex algebraically) to prove the lesser result of the equations for widths.

The equations for W_n -type terms are the following generalizations of equation set (7) in [6] to q -series:

$$G(x, q) = 1 + xqG(xq, q) + \frac{x}{q} G\left(\frac{x}{q}, q\right) + x^2 E(xq, q) G\left(\frac{x}{q}, q\right) \tag{22}$$

$$E(x, q) = S(x, q)(1 + xqE(xq, q)) \tag{23}$$

$$S(x, q) = 1 + x^2 q S(xq, q) S(s, q)(1 + xq^2 E(xq^2, q)) \tag{24}$$

where $G(x, q)$ is the final generating function. The power of x gives the number of sites, and the power of q gives the width of the animal. Note that it differs from the generating functions used previously in this paper by a factor of x due to slightly different conventions used in [6]. E is the generating function for trees that never go right of the root, and S is the generating function for those trees counted in E that return to directly below the root. See [6] for a proof that G is the generating function for directed animals.

The generating functions for $W_n c_n$ can then be obtained from the equation

$$\sum_n c_n W_n^{[i]} x^n = x \left(q \frac{\partial}{\partial q} \right)^i G(x, q). \tag{25}$$

The extra power of x is due to the fact that in [6] the number of sites in the animal is defined so as not to include the initial site, and thus is one fewer than in my definition.

The equations needed for w -type terms are much more complex, requiring $i + 1$ variables to get the i th moment. If the power of x gives the number of sites, the power of m gives the width $w^{[1]}$, and the power of I gives the moment of inertia around the preferred

direction, $w^{[2]}$:

$$G(I, m, x) = 1 + mxIG(I, mI^2, mIx) + m^{-1}xIG(I, mI^{-2}, m^{-1}Ix) \\ + x^2I^2G(I, mI^{-2}, m^{-1}Ix)E(I, mI^2, mIx) \quad (26)$$

$$E(I, m, x) = S(I, m, x)(1 + mxIE(I, mI^2, mIx)) \quad (27)$$

$$S(I, m, x) = 1 + x^2mIS(I, mI^2, mIx)S(I, m, x)(1 + m^2I^4xE(I, mI^4, m^2I^4x)). \quad (28)$$

Again, this set of equations need not be solved in its entirety to get a useful generating function:

$$\sum_n c_n w_n^{[2]} x^n = x \left(\frac{\partial G}{\partial I} \right)_{I=m=1}. \quad (29)$$

Similar sets of equations can be found for the triangular lattice, by adding an extra term to the equations for G and S representing a tree formed by taking the site which is a diagonal bond directly downwards (in the notation used in [6]) from the starting point, but not to the left or right. If a site to the left or right is taken, then the diagonal site's possible occupancy will be automatically taken care of by the existing equations. For just W_n one obtains

$$G(x, q) = 1 + xqG(xq, q) + \frac{x}{q}G\left(\frac{x}{q}, q\right) + x^2E(xq, q)G\left(\frac{x}{q}, q\right) + xG(x, q) \quad (30)$$

$$E(x, q) = S(x, q)(1 + xqE(xq, q)) \quad (31)$$

$$S(x, q) = 1 + x^2qS(xq, q)S(s, q)(1 + xq^2E(xq^2, q)) + xS(x, q) \quad (32)$$

and for both W_n and w_n one obtains

$$G(I, m, x) = 1 + mxIG(I, mI^2, mIx) + m^{-1}xIG(I, mI^{-2}, m^{-1}Ix) \\ + x^2I^2G(I, mI^{-2}, m^{-1}Ix)E(I, mI^2, mIx) + xG(I, m, x) \quad (33)$$

$$E(I, m, x) = S(I, m, x)(1 + mxIE(I, mI^2, mIx)) \quad (34)$$

$$S(I, m, x) = 1 + x^2mIS(I, mI^2, mIx)S(I, m, x)(1 + m^2I^4xE(I, mI^4, m^2I^4x)) \\ + xS(I, m, x). \quad (35)$$

All four empirically derived width formulae were verified via partial derivatives of the above q -series with the help of the computer based algebraic package *Maple*.

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